

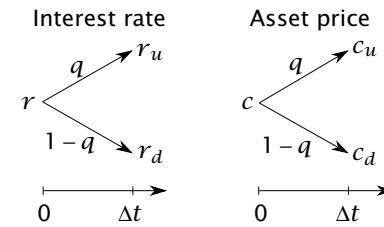
## Interest Rate Models

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B8835

Security Pricing: Models and Computation

### Risk-Neutral Pricing



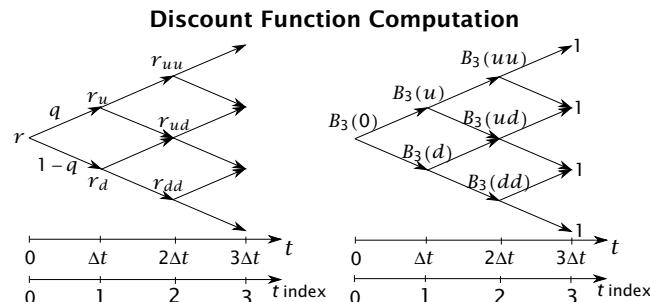
$r$  is the risk-free rate of interest over  $[0, \Delta t]$ . That is,  $r$  is a  $\Delta t$ -period rate and \$1 invested at time 0 will be worth  $e^{r\Delta t}$  at time  $\Delta t$ .

The risk-neutral probability of an upmove is  $q$ . The risk-neutral pricing equation is:

$$c = E[e^{-r\Delta t} c_{\Delta t}] = e^{-r\Delta t} [qc_u + (1 - q)c_d] \quad (1)$$

i.e., the value at time 0 of the random payoff  $c_{\Delta t}$  is the discounted expected payoff under the risk-neutral measure.

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Let  $B_T(x) =$  price in state  $x$  of \$1 paid at time  $T$ . For simplicity, use  $B_i(x) = B_{i\Delta t}(x)$  and  $B_i = B_{i\Delta t}(0)$ . The initial discount function is  $B_i$ , for  $i = 1, 2, \dots, n$ .

Given an interest rate lattice, how is the discount function computed?

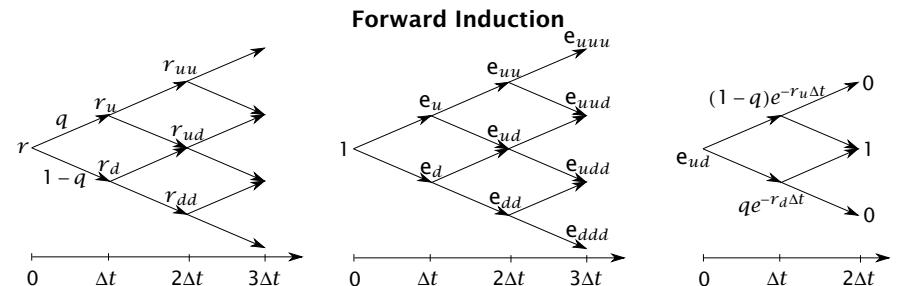
Step 1. Use 1-step lattice to compute  $B_1$ .

Step 2. Use 2-step lattice to compute  $B_2$ .

Step 3. Use 3-step lattice to compute  $B_3$ .

Total work:  $O(n^3)$ .

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Let  $e_x(0) = e_x =$  price at time 0 of \$1 paid in state  $x$  (the *state price* or *Arrow-Debreu price*). The price at time 0 of a discount bond maturing at  $2\Delta t$  is  $B_2 = e_{uu} + e_{ud} + e_{dd}$ .

Step 1.  $e_u = qe^{-r\Delta t}$ ,  $e_d = (1 - q)e^{-r\Delta t} \Rightarrow B_1$

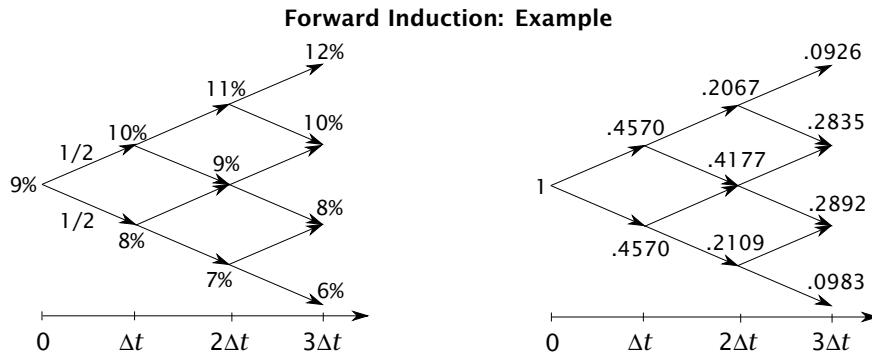
Step 2.  $e_{ud} = (1 - q)e^{-r_u\Delta t}e_u + qe^{-r_d\Delta t}e_d$ ,

$e_{uu} = qe^{-r_u\Delta t}e_u$ ,  $e_{dd} = (1 - q)e^{-r_d\Delta t}e_d \Rightarrow B_2$

Step 3.  $e_{uuu} = qe^{-r_{uu}\Delta t}e_{uu}$ ,  $e_{ddd} = (1 - q)e^{-r_{dd}\Delta t}e_{dd}$

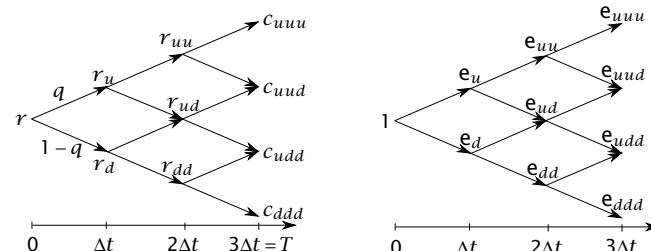
$e_{uud} = (1 - q)e^{-r_{uu}\Delta t}e_{uu} + qe^{-r_{ud}\Delta t}e_{ud}$ , etc.

Total work:  $O(n^2)$ !



The interest rate lattice on the left gives the state prices on the right.  
The initial discount function is:  $B_0 = 1$ ,  $B_1 = 0.9139$ ,  $B_2 = 0.8353$ , and  $B_3 = 0.7636$ . The corresponding initial yield curve is:  $y_1 = 9\%$ ,  $y_2 = 8.998\%$ , and  $y_3 = 8.992\%$ . ( $\Delta t = 1$  year in this example.)

### Pricing European Derivative Securities in a Lattice



Let  $c$  = price at time 0 of  $c_x$  received in state  $x$  at time  $T$ . What is  $c$ ?

Can use backward induction as before, e.g.,

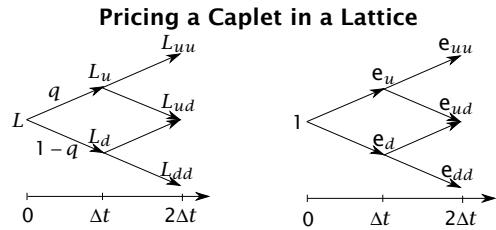
$$c_{uu} = e^{-r_{uu}\Delta t} [q c_{uuu} + (1 - q) c_{uud}],$$

etc. This procedure determines  $c$  in work which is  $O(n^2)$ .

Alternatively, if the state prices have already been computed, then

$$c = \sum_x c_x e_x$$

(the sum is over the  $n + 1$  states  $x$  at time  $T = n\Delta t$ ). Work is  $O(n)$ .



Suppose the lattice is constructed of simply compounded rates and consider a *caplet* for the period  $[2\Delta t, 3\Delta t]$ , struck at  $K$  (with a notional of \$1). If the rate at time  $2\Delta t$  is  $L$ , a payment of  $\max(L - K, 0)\Delta t$  is made at  $3\Delta t$ . The value at  $2\Delta t$  is

$$\frac{\Delta t}{1 + L\Delta t} \max(L - K, 0).$$

So the value of the caplet at time 0 is

$$\sum_{\text{states } x \text{ at } 2\Delta t} \frac{\Delta t}{1 + L(x)\Delta t} \max(L(x) - K, 0) e_x$$

If the continuously compounded lattice rate is  $r$ , convert to  $L$  using

$$e^{-r\Delta t} = \frac{1}{1 + L\Delta t} \quad \text{or} \quad L = \frac{1}{\Delta t} (e^{r\Delta t} - 1).$$

### Equivalence Between Caps on Rates and Puts on Bonds

Consider a *caplet* for the period  $[m\Delta t, (m + 1)\Delta t]$ , struck at  $K$ . The value of the caplet at  $m\Delta t$  when the rate is  $L$  is

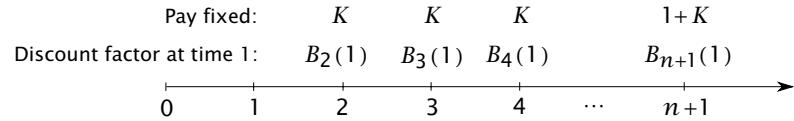
$$\begin{aligned} \frac{\Delta t}{1 + L\Delta t} \max(L - K, 0) &= \max \left( \frac{1 + L\Delta t}{1 + L\Delta t} - \frac{1 + K\Delta t}{1 + L\Delta t}, 0 \right) \\ &= (1 + K\Delta t) \max \left( \frac{1}{1 + K\Delta t} - \frac{1}{1 + L\Delta t}, 0 \right) \end{aligned}$$

i.e., the caplet is equivalent to  $(1 + K\Delta t)$  puts on a one-period discount bond with strike  $\frac{1}{1 + K\Delta t}$  expiring at  $m\Delta t$ .

A caplet is equivalent to an option to enter into a one-period swap.

Since a cap is a sum of caplets, a cap is also equivalent to a portfolio of puts on discount bonds.

### Equivalence Between Swaptions and Options on Coupon Bonds



Consider a 1-year option to enter into an  $n$ -year payer swap (i.e., pay fixed, receive floating) with a strike of  $K$  (and a notional principal of \$1).

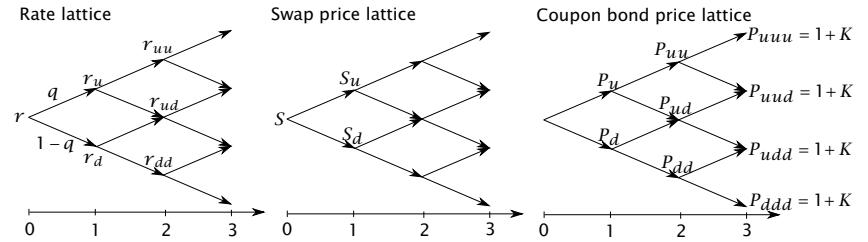
At time 1, the swap is worth  $\text{PV}(\text{floating}) - \text{PV}(\text{fixed})$ . Assuming the principal is exchanged at the end of the swap,  $\text{PV}(\text{floating}) = 1$ . Also,

$$\text{PV}(\text{fixed}) = \sum_{i=2}^{n+1} B_i(1)K + B_{n+1}(1) \equiv P(1),$$

i.e.,  $\text{PV}(\text{fixed}) = \text{value of an } n\text{-year bond with a coupon of } K$ .

So the payoff of the option is  $\max(1 - P(1), 0)$ , i.e., it is equivalent to a put with a strike of 1 on a bond paying a coupon of  $K$ .

### Pricing a European Swaption in a Lattice



What is the value today of a 1-year option to enter into a 2-year payer swap with a strike of  $K$  (and a notional principal of \$1)?

Swap price lattice:  $S_u = \max(1 - P_u, 0)$ ,  $S_d = \max(1 - P_d, 0)$ , and

$$S = \sum_{\text{states } x \text{ at 1}} S_x e_x = e^{-r}(qS_u + (1-q)S_d).$$

Coupon bond price lattice:  $P_{uu} = K + e^{-r_{uu}}(1+K)$ ,  $P_{ud} = K + e^{-r_{ud}}(1+K)$ , ...,  $P_d = e^{-r_d}(qP_{ud} + (1-q)P_{dd})$ .

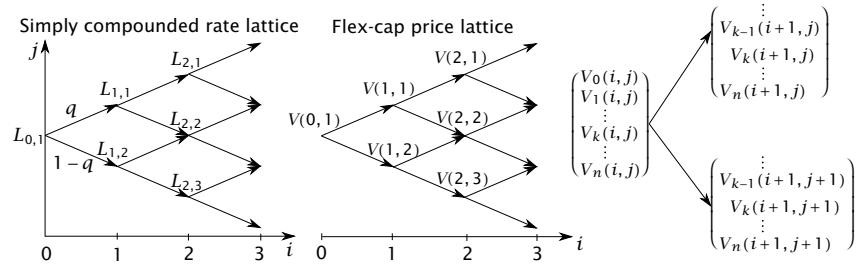
### Flexible Caps

A *flexible cap* gives the holder the right to exercise some (not all) of the caplets comprising the cap. Example: Right to cap at  $K$  at most three out of the next seven quarterly Libor rates.

- *Auto-flex cap*: In-the-money caplets are exercised automatically on fixing dates until none are left.
- *Chooser-flex cap*: On fixing dates, the holder can choose whether to exercise an in-the-money caplet (until none are left).

Auto-flex caps are European-style securities, but are more complicated than plain vanilla caps because they are *path-dependent*. Chooser-flex caps are American-style (holder can choose among various exercise strategies).

### Valuing Chooser-Flex Caps in a Lattice



Need to record a *vector* of information at each node:

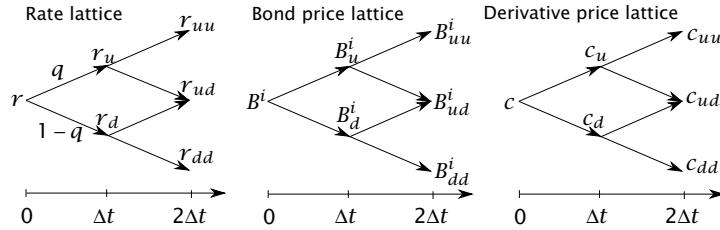
$V_k(i, j)$  = value of chooser-flex cap with strike  $K$  at time  $i$  and rate  $j$  with  $k$  remaining caplets ( $n$  is the maximum number of caplets that can be exercised).

$$V_k(i, j) = \max(\text{don't exercise, exercise}),$$

$$\text{don't} = \frac{1}{1 + L_{i,j}\Delta t} (qV_k(i+1, j) + (1-q)V_k(i+1, j+1))$$

$$\text{exercise} = \frac{L_{i,j} - K}{1 + L_{i,j}\Delta t} + \frac{1}{1 + L_{i,j}\Delta t} (qV_{k-1}(i+1, j) + (1-q)V_{k-1}(i+1, j+1))$$

### Computing Greeks in a Lattice - I

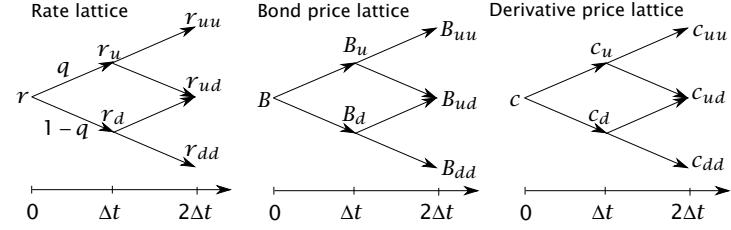


Using bonds  $i = 1, 2$  the derivative can be replicated exactly by solving:

$$\begin{aligned} B_u^1 x_1 + B_u^2 x_2 &= c_u \\ B_d^1 x_1 + B_d^2 x_2 &= c_d \end{aligned}$$

for  $x_1$  and  $x_2$ , where  $x_i$  is the number of units of bond  $i$  in the replicating portfolio,  $i = 1, 2$ . Check that  $B^1 x_1 + B^2 x_2 = c$ !

### Computing Greeks in a Lattice - II



$$\Delta \equiv \frac{\partial c}{\partial B}$$

which can be approximated in a lattice as

$$\Delta \approx \frac{\Delta c}{\Delta B} = \frac{c_u - c_d}{B_u - B_d}.$$

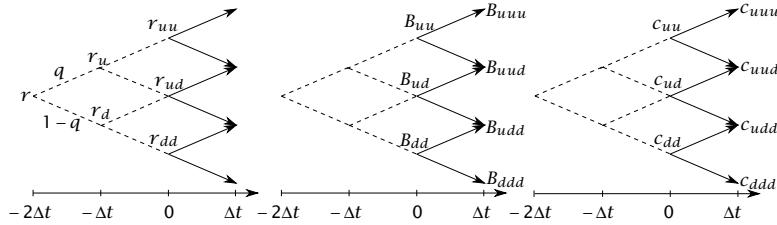
$$\Gamma \equiv \frac{\partial^2 c}{\partial B^2}$$

which can be approximated in a lattice as

$$\Gamma \approx \frac{\Delta u - \Delta d}{B_u - B_d} = \frac{\frac{c_{uu} - c_{ud}}{B_{uu} - B_{ud}} - \frac{c_{ud} - c_{dd}}{B_{ud} - B_{dd}}}{B_u - B_d}.$$

These approximations mix theta ( $\Theta \equiv \partial c / \partial t$ ) with delta and gamma.

### Computing Greeks in a Lattice - III



To avoid mixing theta with delta and gamma, one can alternatively build a lattice extending back two steps, to time  $-2\Delta t$ .

Then

$$\Delta \approx \frac{\Delta c}{\Delta B} = \frac{c_{uu} - c_{dd}}{B_{uu} - B_{dd}}.$$

Similarly,

$$\Gamma \approx \frac{\frac{c_{uu} - c_{ud}}{B_{uu} - B_{ud}} - \frac{c_{ud} - c_{dd}}{B_{ud} - B_{dd}}}{\frac{1}{2}(B_{uu} - B_{dd})}.$$

### Lattice Building

So far we've shown how to price securities given an interest rate lattice. Now we want to build interest rate lattices from scratch. Desirable properties of interest rate lattices:

- Calibrate to market instruments
  - Match the initial term structure
  - Match the initial volatility structure
  - Match prices of other interest rate derivatives
  - Match the initial correlation structure
- Generate reasonable future volatility and correlation structures
- Nonnegative rates
- Convergence of discrete model to continuous-time limit
- Analytical tractability

### Ho-Lee Model

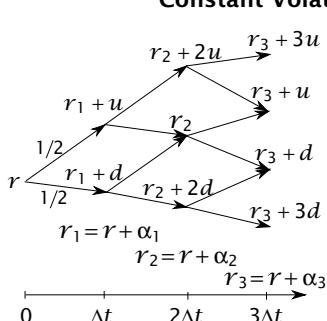
- Match the initial term structure
- Match the initial volatility structure (with the generalized model)
- Short rate is normal
  - Volatility expressed in absolute terms (not as a fraction of current level of rates)
- Rates can go negative
- Single factor model with a Markov process for  $r_t$

The Ho-Lee model (constant volatility):

$$dr_t = b(t)dt + \sigma dW_t$$

The Ho-Lee model (time-varying volatility):

$$dr_t = b(t)dt + \sigma(t)dW_t$$



$$\begin{aligned} e_u &= \frac{1}{2}e^{-r\Delta t}, \quad e_d = \frac{1}{2}e^{-r\Delta t} \\ e_u + e_d &= e^{-r\Delta t} \\ e_{uu} &= \frac{1}{2}e^{-(r_1+u)\Delta t}e_u \\ e_{ud} &= \frac{1}{2}e^{-(r_1+u)\Delta t}e_u + \frac{1}{2}e^{-(r_1+d)\Delta t}e_d \\ e_{dd} &= \frac{1}{2}e^{-(r_1+d)\Delta t}e_d \\ e_{uu} + e_{ud} + e_{dd} &= e^{-(r_1+u)\Delta t}e_u + e^{-(r_1+d)\Delta t}e_d \end{aligned}$$

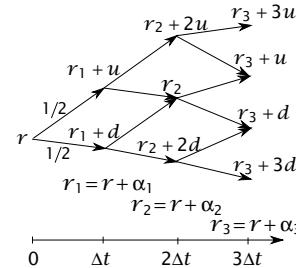
Now choose  $r_{i-1}$  to match the given initial bond price  $B_i$ ,  $i = 1, 2, \dots, n$ .

- Step 1.  $B_1 = e_u + e_d = e^{-r\Delta t}$ . So  $r = -\frac{1}{\Delta t} \ln(B_1)$ . This also gives  $e_u$  and  $e_d$ .  
 Step 2.  $B_2 = e_{uu} + e_{ud} + e_{dd} = e^{-(r_1+u)\Delta t}e_u + e^{-(r_1+d)\Delta t}e_d$ .

$$\Rightarrow r_1 = \frac{\ln(e^{-u\Delta t}e_u + e^{-d\Delta t}e_d) - \ln B_2}{\Delta t}$$

and  $r_1$  gives  $e_{uu}$ ,  $e_{ud}$ , and  $e_{dd}$ . Numerical solution is not needed for  $r_1$ .

### Constant Volatility Ho-Lee Model



$$\begin{aligned} E[X] &= \frac{1}{2}(A+B) \\ \text{Var}[X] &= \frac{1}{4}(A-B)^2 \\ \sigma[X] &= \frac{1}{2}(A-B) \end{aligned}$$

Constant volatility Ho-Lee model:  $dr_t = b(t)dt + \sigma dW_t$ .

Take  $d = -u$ , but how should  $u$  be set?

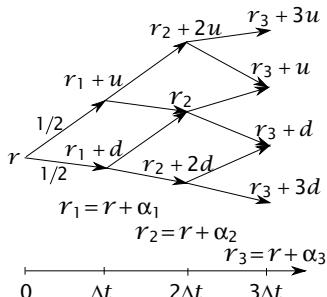
Want  $\text{Var}(r_{\Delta t}) = \sigma^2 \Delta t$ .

$$\text{Var}(r_{\Delta t}) = \frac{1}{4}((r_1 + u) - (r_1 + d))^2 = u^2$$

So,

$$u = \sigma\sqrt{\Delta t}, \quad d = -u.$$

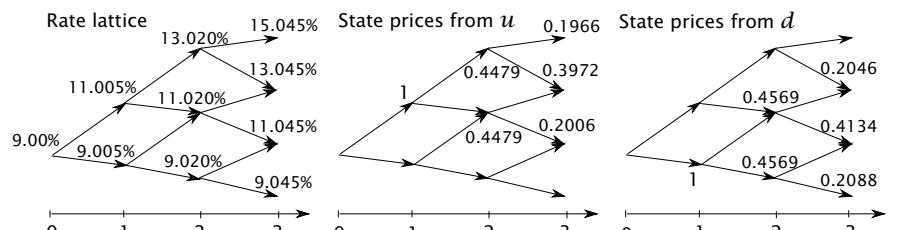
### Constant Volatility Ho-Lee Model



### Ho-Lee Lattice: Numerical Example

Inputs:	Time to maturity	Yield to maturity
	1	9.00%
	2	9.50%
	3	10.00%
	4	10.50%

$$\begin{aligned} u &= 1\% \\ d &= -u \end{aligned}$$



Time	0	1	2	3
Median rate	9.0%	10.005%	11.020%	12.045%

$$\begin{bmatrix} 9.0\% \\ 9.5\% \\ 10.0\% \\ 10.5\% \end{bmatrix} \begin{bmatrix} 11.00\% \\ 11.51\% \\ 12.01\% \end{bmatrix} \begin{bmatrix} 9.00\% \\ 9.51\% \\ 10.01\% \end{bmatrix}$$

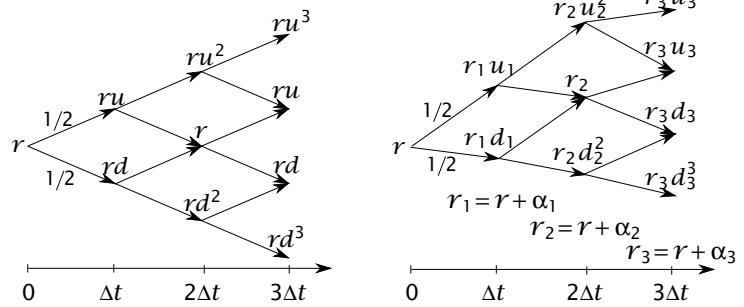
### Black-Derman-Toy Model

- Match the initial term structure
- Match the initial volatility structure
- Short rate is lognormal
  - Volatility expressed as a fraction of current level of rates (relative measure as in Black-Scholes, not absolute)
- Nonnegative rates
- Single factor model with a Markov process for  $r_t$

The BDT model:

$$\frac{dr_t}{r_t} = b(t)dt + \sigma(t)dW_t$$

The functions  $b(t)$  and  $\sigma(t)$  will be determined numerically, i.e., in the lattice construction.



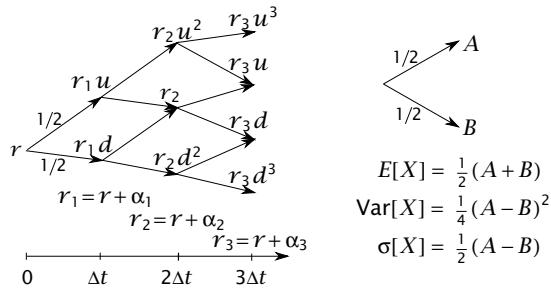
Start with multiplicative (lognormal) lattice.

Shift rates at time  $i\Delta t$  by  $\alpha_i$  to match initial term structure. The median rate at time  $i\Delta t$  is  $r_i = r + \alpha_i$ .

Set volatility parameters  $u_i, d_i$  ( $u_i d_i = 1$ ) at time  $i\Delta t$  to match initial volatility structure.

The BDT lattice is fully specified by  $(r_i, u_i), i = 1, 2, \dots, n$ .

### Constant Volatility BDT Model



Constant volatility BDT model:  $\frac{dr_t}{r_t} = b(t)dt + \sigma dW_t$ .

Take  $d = 1/u$ , but how should  $u$  be set?

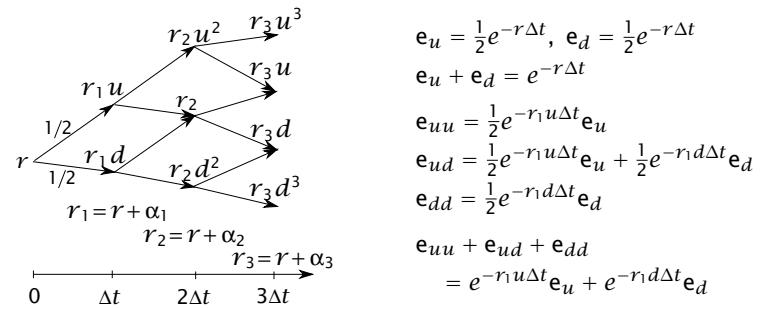
Want  $\text{Var}(\ln(r_{\Delta t})) = \sigma^2 \Delta t$ .

$$\text{Var}(\ln(r_{\Delta t})) = \frac{1}{4}(\ln(r_1 u) - \ln(r_1 d))^2 = \ln(u)^2$$

So,

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = 1/u.$$

### Constant Volatility BDT Model

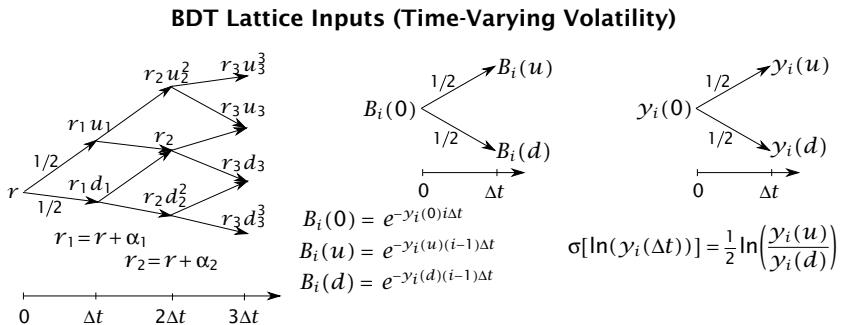


Now choose  $r_{i-1}$  to match the given initial bond price  $B_i$ ,  $i = 1, 2, \dots, n$ .

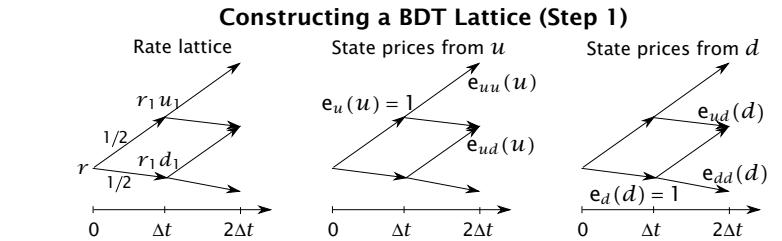
Step 1.  $B_1 = e_u + e_d = e^{-r\Delta t}$ . So  $r = -\frac{1}{\Delta t} \ln(B_1)$ . This also gives  $e_u$  and  $e_d$ .

Step 2.  $B_2 = e_{uu} + e_{ud} + e_{dd} = e^{-r_1 u \Delta t} e_u + e^{-r_1 d \Delta t} e_d$ . Numerical solution gives  $r_1$  and also  $e_{uu}$ ,  $e_{ud}$ , and  $e_{dd}$ .

Step 3.  $B_3 = e^{-r_2 u^2 \Delta t} e_{uu} + e^{-r_2 d \Delta t} e_{ud} + e^{-r_2 d^2 \Delta t} e_{dd}$ . Numerical solution  $r_2$  and also  $e_{uuu}$ ,  $e_{uud}$ ,  $e_{udd}$ , and  $e_{ddd}$ .



(Note  $\sigma_1(0) = 0$ .)



Step 1. Find  $r_1$  and  $u_1$  so that the inputs  $B_2(0)$  and  $\sigma_2(0)$  are matched.

Specifically,  $r_1$  and  $u_1$  give  $B_2(u)$  and  $B_2(d)$  through:

$$e_{uu}(u) = \frac{1}{2}e^{-r_1 u_1 \Delta t}, \quad e_{ud}(u) = \frac{1}{2}e^{-r_1 d_1 \Delta t},$$

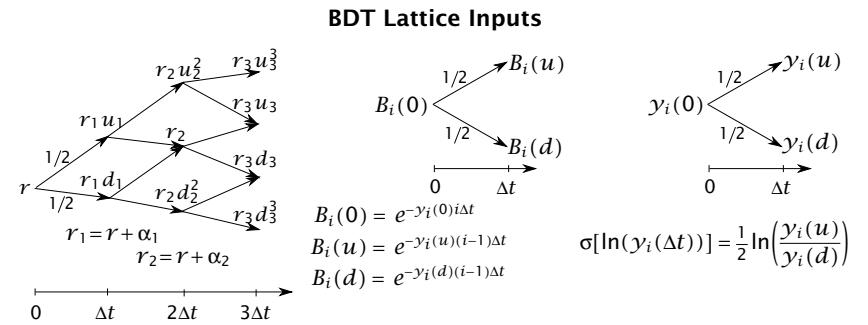
$$e_{ud}(d) = \frac{1}{2}e^{-r_1 d_1 \Delta t}, \quad e_{dd}(d) = \frac{1}{2}e^{-r_1 d_1 \Delta t},$$

$$B_2(u) = e_{uu}(u) + e_{ud}(u), \quad B_2(d) = e_{ud}(d) + e_{dd}(d).$$

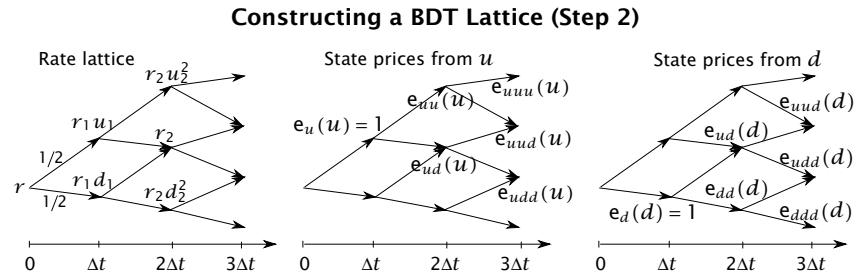
Then  $B_2(u)$  and  $B_2(d)$  give  $B_2(0)$  and  $\sigma_2(0)$  through:

$$B_2(0) = \frac{1}{2}e^{-r\Delta t}(B_2(u) + B_2(d)) \quad \text{and} \quad \sigma_2(0) = \frac{1}{2\sqrt{\Delta t}} \ln\left(\frac{y_2(u)}{y_2(d)}\right).$$

Finding  $r_1$  and  $u_1$  involves solving two nonlinear equations with two unknowns.



Procedure: Choose  $r_i, u_i$  to match  $B_{i+1}(u)$  and  $B_{i+1}(d)$  for  $i = 1, \dots, n - 1$ .



Step 2. Find  $r_2$  and  $u_2$  so that the inputs  $B_3(0)$  and  $\sigma_3(0)$  are matched.

Specifically,  $r_2$  and  $u_2$  give  $B_3(u)$  and  $B_3(d)$  through:

$$B_3(u) = e_{uuu}(u) + e_{uud}(u) + e_{udd}(u), \quad B_3(d) = e_{uud}(d) + e_{udd}(d) + e_{ddd}(d).$$

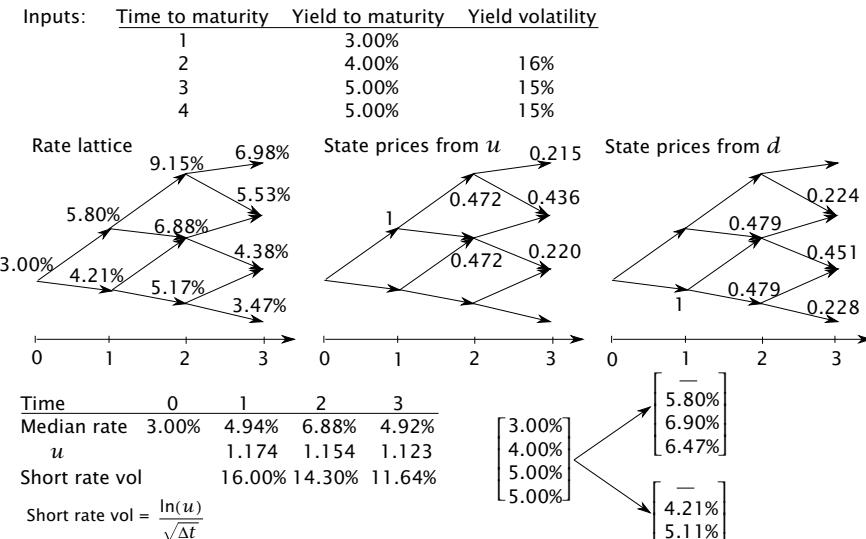
The state prices, e.g.,  $e_{uuu}(u)$ , etc., are determined by forward induction.

Then  $B_3(u)$  and  $B_3(d)$  give  $B_3(0)$  and  $\sigma_3(0)$  through:

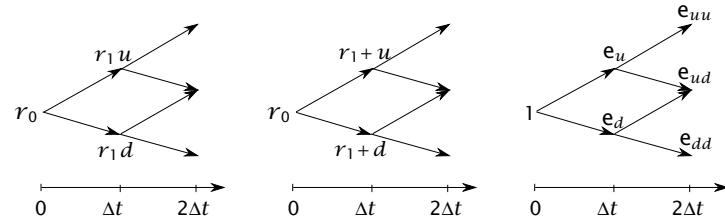
$$B_3(0) = \frac{1}{2}e^{-r\Delta t}(B_3(u) + B_3(d)) \quad \text{and} \quad \sigma_3(0) = \frac{1}{2\sqrt{\Delta t}} \ln\left(\frac{y_3(u)}{y_3(d)}\right).$$

Finding  $r_2$  and  $u_2$  involves solving two nonlinear equations with two unknowns.

### BDT Lattice: Numerical Example



### Comparison of Lognormal BDT with Normal Ho-Lee



BDT with constant volatility, binomial lattice (see p. 25):

$$\begin{aligned} B_2 &= e_{uu} + e_{ud} + e_{dd} \\ &= e^{-r_1 u \Delta t} e_u + e^{-r_1 d \Delta t} e_d. \end{aligned}$$

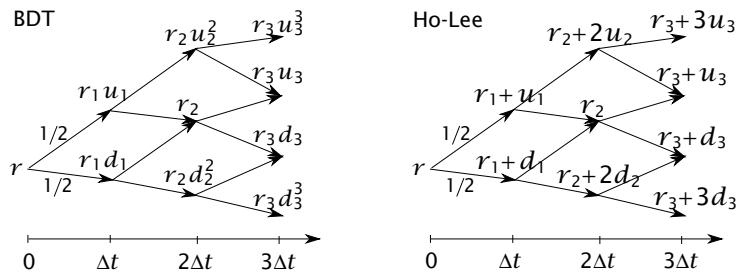
Given  $B_2$ ,  $u$ ,  $d = 1/u$ ,  $e_u$ , and  $e_d$ , still need to solve for  $r_1$  numerically.

Ho-Lee with constant volatility, binomial lattice (see p. 21):

$$\begin{aligned} B_2 &= e_{uu} + e_{ud} + e_{dd} \\ &= e^{-(r_1+u)\Delta t} e_u + e^{-(r_1+d)\Delta t} e_d \\ &= e^{-r_1 \Delta t} (e^{-u \Delta t} e_u + e^{-d \Delta t} e_d) \end{aligned}$$

Can solve for  $r_1$  analytically in terms of  $B_2$ ,  $u$ ,  $d = -u$ ,  $e_u$ , and  $e_d$ .

### Comparison of BDT and Ho-Lee Models



$$\sigma[\ln(\gamma_i(\Delta t))] = \frac{1}{2} \ln\left(\frac{\gamma_i(u)}{\gamma_i(d)}\right)$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$u d = 1$$

$$\frac{dr_t}{r_t} = b(t)dt + \sigma(t)dW_t$$

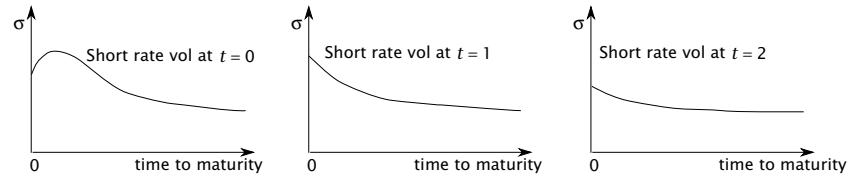
$$\sigma[\gamma_i(\Delta t)] = \frac{1}{2} (\gamma_i(u) - \gamma_i(d))$$

$$u = \sigma\sqrt{\Delta t}$$

$$u + d = 0$$

$$dr_t = b(t)dt + \sigma(t)dW_t$$

### BDT Model with Time-Varying Volatility



Long-term rates are typically less volatile than short rates. Building a BDT lattice consistent with this observation (or consistent with a sequence of cap or swaption prices) requires the short rate volatility  $\sigma$  to vary with time (i.e., non-constant  $u_i$ 's in the lattice).

In this case, the short rate volatility will "walk the volatility curve" and future volatility curves can look quite different than today's.

## Catalog of Spreadsheets

- HL.C.XLS
    - Ho-Lee model with constant volatility (10-step lattice)
    - Matches the initial yield curve automatically
  - HL.XLS
    - Ho-Lee model with time-varying volatility (10-step lattice)
    - Uses the Solver in Excel to match the initial yield curve and initial volatility curve
  - BDT.XLS
    - Black-Derman-Toy model with time-varying volatility (10-step lattice)
    - Uses the Solver in Excel to match the initial yield curve and initial volatility curve
  - BDT.C.XLS
    - Black-Derman-Toy model with constant volatility (10-step lattice)
    - Uses the Solver in Excel to match the initial yield curve

HL-C.XLS

## Catalog of Spreadsheets (continued)

- HW.XLS
    - Hull-White model (10-step lattice)
    - Matches the initial yield curve automatically
  - HJM\_SIMN.XLS
    - Normal HJM simulation model (10-step simulation, 3 factor)
    - Simulation can be run with the Excel add-in Crystal Ball
  - HJM\_SIML.XLS
    - Pseudo-lognormal HJM simulation model (10-step simulation, 3 factor)
    - Simulation can be run with the Excel add-in Crystal Ball
  - FACTOR.XLS
    - Computes correlation matrix corresponding to 3-factor HJM model

Note: All spreadsheets use continuously compounded rates.

HL,XLS

BDT.XLS (Sheet 1: rates)

User input in bold italic (T)

User input in bold italic (Initial yield and volatility):

Solver decision variables in box

**BDT.XLS (Sheet 3: state prices)**

BDT.XLS (Sheet 4: state prices in up state)

BDT.XLS (Sheet 5: state prices in down state)

**BDT\_C.XLS**

A	B	C	D	E	F	G	H	I	J	K	L
1	BDT_C.XLS	Black-Derman-Toy Model									
2	Created by Mark Broadie and Paul Glasserman	Constant volatility (10-step lattice)									
3	T	30	nt	dt	10	u(i)=1/d(i)	1.1891				
4	sig	10.00%			3						
5											
6	Time	0	3	6	9	12	15	18	21	24	27
7	Initial yield	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%
8	Median rate	10.00%	9.90%	9.86%	9.96%	10.13%	10.40%	10.76%	11.22%	11.79%	12.46%
9	Initial price	0.7408	0.5488	0.4066	0.3012	0.2231	0.1653	0.1225	0.0907	0.0672	0.0498
10	Price from lattice	0.7408	0.5488	0.4066	0.3012	0.2231	0.1653	0.1225	0.0907	0.0672	0.0498
11	Price_up	1.0000	0.7026	0.4922	0.3440	0.2400	0.1673	0.1166	0.0813	0.0567	0.0396
12	Price_dn	1.0000	0.7791	0.6055	0.4692	0.3624	0.2790	0.2140	0.1636	0.1247	0.0948
13	Yield_up	0.1177	0.1182	0.1186	0.1192	0.1194	0.1195	0.1196			
14	Yield_dn	0.0832	0.0836	0.0841	0.0846	0.0851	0.0856	0.0862	0.0867		
15	Yield volatility from lattice	0.1000	0.0998	0.0992	0.0984	0.0973	0.0959	0.0944	0.0927	0.0908	
16											
17	Rates:	0	1	2	3	4	5	6	7	8	9
18	10										
19	9										
20	8										
21	7										
22	6										
23	5										
24	4										
25	3										
26	2										
27	1										
28	0	10.00%	9.88%	10.13%	10.76%	11.79%	11.22%	11.79%	12.46%	12.46%	12.46%
29	-1	8.32%	8.38%	8.74%	9.44%	10.48%					
30	-2	6.99%	7.16%	7.61%	8.34%	9.34%	10.34%	11.34%	12.34%	13.34%	14.34%
31	-3	5.92%	6.18%	6.67%	7.41%						
32	-4										
33	-5										
34	-6										
35	-7										
36	-8										
37	-9										
38	-10										

**HW.XLS**

A	B	C	D	E	F	G	H	I	J	K	L
1	HW.XLS	Hull-White Model									
2	Created by Mark Broadie and Paul Glasserman	(10-step lattice)									
3	T	5	nt	dt	10	m	-0.05				
4	a	0.1	dt	0.5	v	0.0005					
5	sig	0.01	dr	1.22%	u_max	4					
6											
7	Time	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
8	Initial yield	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%
9	Median rate	10.00%	10.00%	10.01%	10.02%	10.03%	10.04%	10.05%	10.06%	10.07%	
10	Initial price	0.9512	0.9048	0.8607	0.8187	0.7788	0.7408	0.7047	0.6703	0.6376	0.6065
11	Price from lattice	0.9512	0.9048	0.8607	0.8187	0.7788	0.7408	0.7047	0.6703	0.6376	0.6065
12	Price_up	1.0000	0.9454	0.8941	0.8458	0.8003	0.7574	0.7170	0.6790	0.6431	0.6092
13	Price_md	1.0000	0.9512	0.9048	0.8607	0.8187	0.7787	0.7407	0.7045	0.6701	0.6374
14	Price_dn	1.0000	0.9571	0.9157	0.8758	0.8375	0.8006	0.7651	0.7310	0.6983	0.6669
15	Yield_up	0.1123	0.1120	0.1117	0.1114	0.1111	0.1109	0.1106	0.1104	0.1101	0.1101
16	Yield_md	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
17	Yield_dn	0.0878	0.0881	0.0884	0.0887	0.0890	0.0892	0.0895	0.0898	0.0900	0.0902
18	Yield volatility from lattice	0.0100	0.0098	0.0095	0.0093	0.0090	0.0088	0.0086	0.0084	0.0082	
19											
20	Rates:	0	1	2	3	4	5	6	7	8	9
21	10										
22	9										
23	8										
24	7										
25	6										
26	5										
27	4										
28	3										
29	2										
30	1										
31	0	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%
32	-1	8.78%	8.78%	8.78%	8.78%	8.78%	8.78%	8.78%	8.78%	8.78%	8.78%
33	-2	7.56%	7.56%	7.56%	7.56%	7.56%	7.56%	7.56%	7.56%	7.56%	7.56%
34	-3										
35	-4										
36	-5										
37	-6										
38	-7										
39	-8										
40	-9										
41	-10										

**HJM\_SIMN.XLS**

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	HJM_SIMN.XLS	Heath-Jarrow-Morton Simulation Model											
2	Created by Mark Broadie and Paul Glasserman	Three-factor normal version											
3	T	20	nt	dt	2								
4	nt	10	sqr(dt)	1.4142									
5													
6	Time index	0	1	2	3	4	5	6	7	8	9	10	11
7	Time	0	2	4	6	8	10	12	14	16	18	20	22
8	scale	Initial yield	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%
9	1.0000	Factor 1	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%
10	1.0000	Factor 2	0.67%	0.45%	0.30%	0.20%	0.14%	0.09%	0.06%	0.04%	0.03%	0.02%	
11	1.0000	Factor 3	0.40%	0.30%	0.20%	0.10%	0.06%	-0.10%	-0.20%	-0.30%	-0.40%	-0.50%	
12													
13	Time index	Time	Spot rate	Forward rates									
14	0	0.00	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%
15	1	2.00	9.50%	10.00%	10.42%	10.69%	11.05%	11.19%	11.32%	11.43%	11.54%		
16	2	4.00	7.67%	8.33%	8.88%	9.23%	9.49%	9.61%	9.81%	9.93%	10.05%		
17	3	6.00	8.24%	9.01%	9.50%	9.90%	10.16%	10.37%	10.54%	10.68%			
18	4	8.00	6.64%	7.79%	8.61%	9.21%	9.69%	10.05%	10.38%				
19	5	10.00	7.04%	8.05%	8.86%	9.53%	10.13%	10.68%					
20	6	12.00	9.40%	10.28%	11.01%	11.68%	12.32%						
21	7	14.00	14.51%	15.28%	15.98%	16.65%							
22	8	16.00	12.82%	13.52%	14.18%								
23	9	18.00	14.81%	14.98%									
24	10	20.00	16.14%										
25													

**HJM\_SIML.XLS (Sheet1: Forward curves)**

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	HJM_SIML.XLS	Heath-Jarrow-Morton Simulation Model											
2	Created by Mark Broadie and Paul Glasserman	Three-factor pseudo-lognormal version											
3	T	5	nt	dt	0.5								
4	nt	10	sqr(dt)	0.7071									
5													
6	Time index	0	1	2	3	4	5	6	7</				

**HJM\_SIML.XLS (Sheet of Discount Bond Prices)**

A	B	C	D	E	F	G	H	I	J	K	L	M
1		Discount bond price										
2	Time index	Path price	True price									
3	1	0.9512										
4	2	0.8988	0.9048									
5	3	0.8514	0.8607									
6	4	0.8017	0.8187									
7	5	0.7557	0.7788									
8	6	0.7127	0.7408									
9	7	0.6722	0.7047									
10	8	0.6340	0.6703									
11	9	0.6024	0.6376									
12	10	0.5745	0.6065									
13	11	0.5517	0.5769									
14	Time index	0	1	2	3	4	5	6	7	8	9	10
15	Time	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
16	Time index	Time	Discount bond price									
17	0	0.00	0.9512	0.9048	0.8607	0.8187	0.7788	0.7408	0.7047	0.6703	0.6376	0.6065
18	1	0.50	0.9449	0.8927	0.8433	0.7967	0.7526	0.7109	0.6715	0.6343	0.5991	0.5659
19	2	1.00	0.9473	0.8973	0.8501	0.8054	0.7631	0.7232	0.6854	0.6496	0.6158	
20	3	1.50	0.9416	0.8867	0.8350	0.7864	0.7407	0.6977	0.6573	0.6193		
21	4	2.00	0.9427	0.8895	0.8400	0.7941	0.7512	0.7113	0.6739			
22	5	2.50	0.9430	0.8899	0.8403	0.7938	0.7504	0.7097				
23	6	3.00	0.9433	0.8901	0.8402	0.7933	0.7492					
24	7	3.50	0.9432	0.8899	0.8400	0.7932						
25	8	4.00	0.9502	0.9027	0.8575							
26	9	4.50	0.9536	0.9087								
27	10	5.00	0.9604									

**HJM\_SIML.XLS (Sheet of Factor 1 Drifts)**

A	B	C	D	E	F	G	H	I	J	K	L	M
1												
2	dt	0.5										
3												
4	Time index	0	1	2	3	4	5	6	7	8	9	10
5	Time	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
6												
7	Time index	Time	Drifts for factor 1									
8	0	0.00	0.0025%	0.0075%	0.0125%	0.0175%	0.0225%	0.0275%	0.0325%	0.0375%	0.0425%	0.0475%
9	1	0.50	0.0032%	0.0097%	0.0162%	0.0227%	0.0292%	0.0357%	0.0422%	0.0487%	0.0552%	
10	2	1.00	0.0029%	0.0088%	0.0146%	0.0204%	0.0262%	0.0319%	0.0375%	0.0432%		
11	3	1.50	0.0036%	0.0108%	0.0180%	0.0252%	0.0323%	0.0393%	0.0464%			
12	4	2.00	0.0034%	0.0099%	0.0161%	0.0221%	0.0278%					
13	5	2.50	0.0034%	0.0100%	0.0163%	0.0226%	0.0286%					
14	6	3.00	0.0034%	0.0100%	0.0166%	0.0231%						
15	7	3.50	0.0034%	0.0100%	0.0166%							
16	8	4.00	0.0026%	0.0079%								
17	9	4.50	0.0023%									
18	10	5.00										

**FACTOR.XLS**

A	B	C	D	E	F	G	H	I	J	K	L	
1	FACTOR.XLS	Compute correlation matrix corresponding to the three factors in the HJM model										
2	Created by Mark Broadie and Paul Glasserman											
3	Time index	0	1	2	3	4	5	6	7	8	9	10
4	Factor 1	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	
5	Factor 2	9.05%	8.19%	7.41%	6.70%	6.07%	5.49%	4.97%	4.49%	4.07%	3.68%	
6	Factor 3	2.00%	1.50%	1.00%	0.50%	0.00%	-0.50%	-1.00%	-1.50%	-2.00%	-2.50%	
7												
8	Total volatility	13.63%	13.01%	12.49%	12.05%	11.70%	11.42%	11.21%	11.07%	10.98%	10.94%	
9	Weight 1	0.7335	0.7686	0.8009	0.8299	0.8550	0.8758	0.8921	0.9037	0.9109	0.9137	
10	Weight 2	0.6637	0.6293	0.5934	0.5563	0.5186	0.4807	0.4430	0.4061	0.3703	0.3361	
11	Weight 3	0.1467	0.1153	0.0801	0.0415	0.0000	-0.0438	-0.0892	-0.1356	-0.1822	-0.2284	
12	Correlation matrix	1	2	3	4	5	6	7	8	9	10	
13		1	1.0000	0.9983	0.9930	0.9841	0.9713	0.9550	0.9352	0.9125	0.8872	0.8598
14		2		1.0000	0.9982	0.9927	0.9835	0.9706	0.9541	0.9345	0.9121	0.8874
15		3			1.0000	0.9981	0.9925	0.9832	0.9702	0.9539	0.9347	0.9130
16		4				1.0000	0.9981	0.9924	0.9831	0.9703	0.9544	0.9358
17		5					1.0000	0.9981	0.9925	0.9833	0.9709	0.9555
18		6						1.0000	0.9981	0.9926	0.9837	0.9718
19		7							1.0000	0.9982	0.9929	0.9844
20		8								1.0000	0.9982	0.9932
21		9									1.0000	0.9983
22		10										1.0000

**References**

- o Jamshidian, F., 1991, "Forward induction and construction of yield curve diffusion models," *Journal of Fixed Income*, June, 62–74.
- o Black, F., E. Derman, and W. Toy, 1990, "A one-factor model of interest rates and its application to Treasury bond options," *Financial Analysts Journal*, Vol.46, 33–39.
- o Hull, J., 2000, *Options, Futures, and Other Derivatives*, 4th edition, Prentice-Hall, Upper Saddle River, NJ.
- o Rebonato, R., 1998, *Interest-Rate Option Models*, 2nd edition, John Wiley & Sons, Chichester, England.