Understanding $N(d_1)$ and $N(d_2)$: Risk-Adjusted Probabilities in the Black-Scholes Model [1]

Lars Tyge Nielsen
INSEAD
Boulevard de Constance
77305 Fontainebleau Cedex
France
E-mail: nielsen@freiba51

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Abstract

This paper uses risk-adjusted lognormal probabilities to derive the Black-Scholes formula and explain the factors $N(d_1)$ and $N(d_2)$. It also shows how the one-period and multi-period binomial option pricing formulas can be restated so that they involve analogues of $N(d_1)$ and $N(d_2)$ which have the same interpretation as in the Black-Scholes model.

Cet article utilise les probabilités lognormaux corrigées du risque pour dériver la formule de Black-Scholes et expliquer les facteurs $N(d_1)$ et $N(d_2)$. Il montre aussi comment les modèles binomiaux des prix d'options d'une et de plusieurs périodes peuvent être exprimés d'une façon telle qu'ils impliquent des analogues de $N(d_1)$ et $N(d_2)$ qui ont la même interprétation que dans le modèle de Black-Scholes.

1 Introduction

The Black-Scholes formula is an expression for the current value of a European call option on a stock which pays no dividends before expiration of the option. The formula expresses the call value as the current stock price times a probability factor $N(d_1)$, minus the discounted exercise payment times a second probability factor $N(d_2)$.

Explaining $N(d_1)$ and $N(d_2)$, and in particular explaining why they are different from each other, usually presents some difficulties. Among the major research papers, Black and Scholes (1973) did not explain or interpret $N(d_1)$ and $N(d_2)$. Neither did Merton (1973, 1990 Chapter 8), Cox and Ross (1976), or Rubinstein (1976). As for the textbooks, Jarrow and Rudd (1983) heuristically derive the Black-Scholes formula using risk-adjusted probabilities, and in the process they do interpret $N(d_1)$ and $N(d_2)$. Cox and Rubinstein (1985) state that the stock price times $N(d_1)$ is the present value of receiving the stock if and only if the option finishes in the money, and the discounted exercise payment times $N(d_2)$ is the present value of paying the exercise price in that event. They do not explain why this is so or relate it to the probability that the option finishes in the money. Hull (1989) and Hull (1991) do not explain $N(d_1)$ and $N(d_2)$, although the necessary mathematics is available in the earlier book.

The purpose of the present paper is to explain where $N(d_1)$ and $N(d_2)$ come from and why they are different from each other. This is done by relating them to risk-adjusted probabilities in both the Black-Scholes and in the binomial model of Cox, Ross and Rubinstein (1979). The argument relating to Black-Scholes expands on that of Jarrow and Rudd (1983). The comments on the binomial model involve simple manipulations and reinterpretations of well-known formulas.

Briefly stated, $N(d_2)$ is the risk-adjusted probability that the option will be exercised. The interpretation of $N(d_1)$ is a bit more complicated. The expected value, computed using risk-adjusted probabilities, of receiving the stock at expiration of the option, contingent upon the option finishing in the money, is $N(d_1)$ multiplied by the current stock price and the riskless compounding factor. Thus, $N(d_1)$ is the factor by which the present value of contingent receipt of the stock exceeds the current stock price.

The present value of contingent receipt of the stock is not equal to but larger than the current stock price multiplied by $N(d_2)$, the risk-adjusted probability of exercise. The reason for this is that the event of exercise is not independent of the future stock price. If exercise were completely random and unrelated to the stock price, then indeed the present value of contingent receipt of the stock would be the current stock price multiplied by $N(d_2)$. Actually the present value is larger than this, since exercise is dependent on the future stock price and indeed happens when the stock price is high.

The organization of the paper is as follows. Section 2 states the Black-Scholes formula. Section 3 contains the substance of the argument. It splits the payoff to the call option into two components, shows how their future expected values (computed using the risk-adjusted probabilies) and present values involve the probability factors $N(d_1)$ and $N(d_2)$, and explains why $N(d_1)$ is larger than $N(d_2)$. Section 4 shows how the one-period binomial option pricing formula can be restated in a form which resembles the Black-Scholes formula. It involves analogues of $N(d_1)$ and $N(d_2)$ with similar interpretations as in the Black-Scholes model. Section 5 does the same analysis of the multiperiod binomial model. The rest of the paper contains the documentation to back up the Black-Scholes model: Section 6 explains the probabilistic assumptions behind the model, Section 7 describes how the risk-adjustment of the probabilities is carried out, and Section 8 uses the risk-adjusted probabilities to derive the Black-Scholes formula by computing the present values of the components of the call option payoff. Section 9 contains the conclusion.

2 The Black-Scholes Formula

The Black-Scholes formula is an expression for the current value of a European call option on a stock which pays no dividends before expiration of the option. The formula is

$$C = SN(d_1) - e^{-r\tau} XN(d_2),$$

where C is the current value of the call, S is the current value of the stock, r is the interest rate (assumed constant), τ is the remaining time to expiration of the option, X is the exercise price, and $N(d_1)$ and $N(d_2)$ are probability factors: N is the cumulative standard normal distribution function,

$$d_2 = -\frac{\log(X/S) - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},$$

$$d_1 = d_2 + \sigma\sqrt{\tau},$$

and σ is a parameter measuring the volatility of the stock (interpreted more precisely below in Section 6).

3 The Payoff and Value of the Call

To value the call option, I shall use the concept of risk-adjusted probabilities. It turns out that one can adjust the probability distribution of the stock price in such a way that the current value of any stock-price contingent claim equals the expected future payoff to the claim, computed using the adjusted probabilities, discounted at the riskless rate.

The demonstration of this involves mathematical analysis of dynamic trading and arbitrage between the stock and the riskless asset, something I shall not focus on here. However, the practical aspect of exactly how the probability adjustment is performed, is described below in Section 7.

The payoff to the call option at maturity T will be

$$C_T = \max\{0, S_T - X\} = \begin{cases} S_T - X & \text{if } S_T \geq X \\ 0 & \text{otherwise} \end{cases}$$

It is useful to split this payoff into two components. The first component is the payment of the exercise price, contingent on the option finishing in the money. It will be referred to as "contingent exercise payment," for short. It is a claim with payoff

$$C_T^1 = \begin{cases} -X & \text{if } S_T \geq X \\ 0 & \text{otherwise} \end{cases}$$

The second component is the receipt of the stock, again contingent on the option finishing in the money. It will be referred to as "contingent receipt of the stock. The payoff is

$$C_T^2 = \begin{cases} S_T & \text{if } S_T \geq X \\ 0 & \text{otherwise} \end{cases}$$

The various payoffs are shown in Figure 1.

We can value the option by valuing each of the two components separately. The current value of the contingent payment of the exercise price will be the expected future payment, computed on the basis of the risk-adjusted probability distribution, discounted at the riskless rate. The expected future payoff is

$$EC_T^1 = -XP\{S_T > X\}.$$

where P is the risk-adjusted probability, and so the value is

$$-e^{-r\tau}XP\{S_T > X\}.$$

It turns out that the risk-adjusted probability of the event that the option will finish in the money is $P\{S_T > X\} = N(d_2)$. Therefore, the expected "payoff" is $-XN(d_2)$, and the present discounted value of this payoff is

$$-e^{-r\tau}XN(d_2).$$

So, this is the current value of the first component of the option, the contingent exercise payment.

The current value of the second component of the option, the contingent receipt of the stock, will also equal the expected future value, computed using the adjusted probabilities, and discounted at the riskless rate. The expected

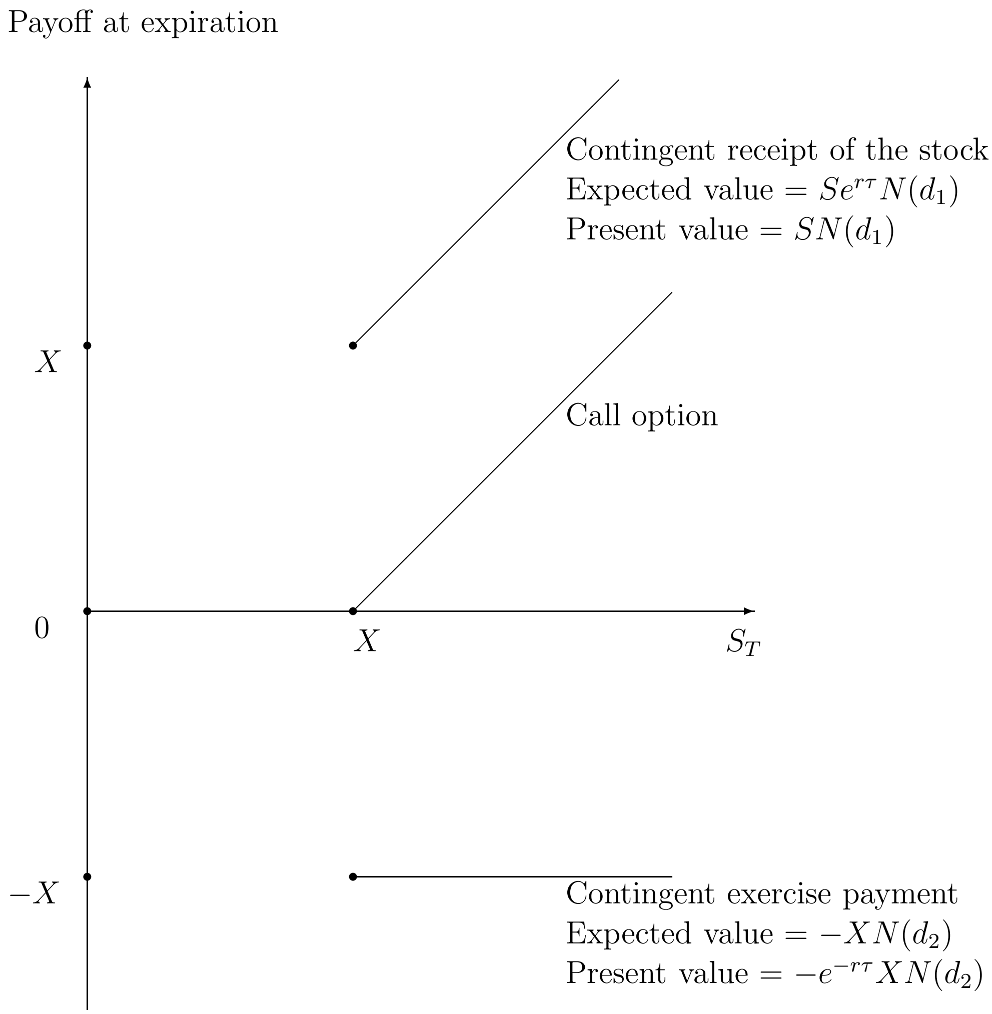

Figure 1: Payoff to the call and its components

future value of this component of the payoff is not simply the conditional expectation of the stock price given exercise. Rather, it is the conditional expectation of the stock price given exercise times the probability of exercise,

$$EC_T^2 = E[S_T | S_T > X] P\{S_T > X\}.$$

It turns out that this equals

$$E[S_T | S_T > X] P\{S_T > X\} = e^{r\tau} S N(d_1),$$

and so the current value is

$$SN(d_1).$$

So, $N(d_1)$ is the factor by which the discounted expected value of contingent receipt of the stock exceeds the current value of the stock.

By putting together the values of the two components of the option payoff, we get the Black-Scholes formula:

$$C = SN(d_1) - e^{-r\tau} X N(d_2).$$

Why is the present value of the contingent receipt of the stock not $SN(d_2)$, corresponding to an expected future value (computed using risk-adjusted probabilities) of $e^{r\tau} SN(d_2)$?

The argument would be this. The present value of unconditionally receiving the stock at time T is obviously equal S, the current stock value. Therefore, the expected future value of unconditionally receiving the stock has to be $Se^{r\tau}$. Now if the stock is received not unconditionally but conditionally on an event which has probability $N(d_2)$, then the expected value should be $Se^{r\tau} N(d_2)$ and the present value should be $SN(d_2)$.

In fact, the present value of contingent receipt of the stock is strictly larger than $SN(d_2)$: since $d_1 > d_2$, it must be the case that $SN(d_1) > SN(d_2)$.

If the present value were equal to $SN(d_2)$, then the value of the call option would be $(S - e^{-r\tau} X) N(d_2)$. This would be negative when the option is out of the money, which clearly cannot be the case.

The error in the argument is that the event of exercise is not independent of the random magnitude of S_T. If exercise were completely random and unrelated to the stock price, then indeed the present value of contingent receipt of the stock would be $SN(d_2)$. Actually, exercise is not purely random but depends on the future stock price: it happens only when S_T is high. Therefore, $Se^{r\tau}N(d_2)$ underestimates the expected value.

Indeed, the expected future value of contingent receipt of the stock is

$$\begin{aligned} E[S_T|S_T > X]P\{S_T > X\} &= E[S_T|S_T > X]N(d_2) \\ &> Se^{r\tau}N(d_2) \end{aligned}$$

because the correlation between S_T and the exercise decision implies that

$$E[S_T|S_T > X] > Se^{r\tau}.$$

4 The One-Period Binomial Model

This section shows how the one-period binomial option pricing formula can be rewritten so as to resemble the Black-Scholes formula, and identifies the analogues of $N(d_1)$ and $N(d_2)$.

In the binomial model, the stock price will either go up by a factor of u from S to uS or down by a factor of d from S to dS. There is a riskless asset, for example a riskless bond, whose price will in any case go up by a factor of r^* (so that r^{*-1} is the riskless discount factor). To avoid a situation where one of the assets dominates the other, it is assumed that $d < r^* < u$ and that the actual probabilities of both an up movement and a down movement in the stock price are positive (although it does not matter how large those probabilities are).

The absence of riskless arbitrage opportunities implies that the current value C of a call option on the stock, with exercise price X, maturing at the end of the period, must be

$$C = r^{*-1}[pC_u + (1-p)C_d],$$

where

$$p = \frac{r^* - d}{u - d}$$

is the risk-adjusted probability that the stock price will go up, C_u is the future value of the option if the stock price goes up, and C_d is the future value of the option if the stock price goes down.

Consider the case where $dS < X < uS$. The option will be exercised if the stock price goes up, and it will expire worthless if the stock price goes down. So, $C_u = uS - X$ and $C_d = 0$. We can rewrite the call price formula like this:

$$\begin{aligned} C &= r^{*-1}[pC_u + (1-p)C_d] \\ &= r^{*-1}p(uS - X) \\ &= [r^{*-1}pu]S - r^{*-1}pX. \end{aligned}$$

This formula is analogous to the Black-Scholes formula. The factor $r^{*-1}pu$ corresponds to $N(d_1)$. It equals the factor by which the discounted expected value of contingent receipt of the stock exceeds the current value of the stock. The discount factor r^{*-1} corresponds to the discount factor $e^{-r\tau}$ in the Black-Scholes model. The factor p in front of X corresponds to $N(d_2)$; it is the risk-adjusted probability of exercise.

A warning: In the Black-Scholes model, $N(d_1)$ equals the hedge ratio, that is, the number of stock to be included in a portfolio of stocks and bonds which replicates the option. It is not true in the binomial model that the analogous factor

$$r^{*-1}pu = r^{*-1}\frac{r^* - d}{u - d}u$$

equals the hedge ratio. In fact, in the present case where $dS < X < uS$, the hedge ratio is

$$H = \frac{C_u - C_d}{(u-d)S} = \frac{uS - X}{(u-d)S},$$

which equals the factor above only in the exceptional case where

$$X = r^{*-1}udS.$$

If $X \leq dS$, then the option will always be exercised, so $C_u = uS - X$ and $C_d = dS - X$. The call price formula can be rewritten as

$$\begin{aligned} C &= r^{*-1}[pC_u + (1-p)C_d] \\ &= r^{*-1}[pu + (1-p)d]S - r^{*-1}X \\ &= S - r^{*-1}X. \end{aligned}$$

Again, the formula is analogous to the Black-Scholes formula. The risk-adjusted probability of exercise, which corresponds to $N(d_2)$, is one. The factor by which the discounted expected value of contingent receipt of the stock exceeds the current value of the stock, which corresponds to $N(d_1)$, is also one.

If $uS \leq X$, then the option will never be exercised, and $C_u = C_d = 0$. The current value of the call will be zero, which is again analogous with Black-Scholes. The risk-adjusted probability of exercise, which corresponds to $N(d_2)$, is zero. The factor by which the discounted expected value of contingent receipt of the stock exceeds the current value of the stock, which corresponds to $N(d_1)$, is also zero.

5 The Multi-Period Binomial Model

This section shows that also the multi-period binomial option pricing formula can be interpreted like the Black-Scholes formula, with factors that formally correspond to $N(d_1)$ and $N(d_2)$ and have the same interpretation.

Let us say there are n periods. In each period, the stock price will either go up by a factor of u or down by a factor of d, and the price of the riskless asset will in any case go up by a factor of r^*. In every period, the actual probability of an up movement of the stock price is positive, and so is the probability of a down movement; but the magnitude of these probabilities is unimportant, and they do not have to stay constant from period to period.

The absence of riskless arbitrage opportunities implies that the current value C of a call option on the stock, with exercise price X, maturing at the end of the period, must be

$$C = S\Phi[a; n, \hat{p}] - Kr^{*-n}\Phi[a; n, p],$$

where a is the minimum number of upward moves necessary for the stock to finish in the money,

$$p = \frac{r^* - d}{u - d}$$

is the risk-adjusted probability that the stock price will go up (in any one period),

$$\hat{p} = r^{*-1}up,$$

and Φ is the complementary binomial distribution function:

$$\Phi[a; n, p] = \sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!}\right) p^j (1-p)^{n-j}.$$

The formula for C is analogous to the Black-Scholes formula. The factor $\Phi[a; n, p]$ is the binomial risk-adjusted probability that the stock price will go up at least a times so that the option will be exercised, and that corresponds to $N(d_2)$. The factor $\Phi[a; n, \hat{p}]$ can also be rewritten so as to correspond to $N(d_1)$ in the Black-Scholes formula:

$$\Phi[a; n, p] \quad = \quad \sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!}\right) \hat{p}^j (1-\hat{p})^{n-j}$$

$$= \quad r^{*-n} \sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!} \right) p^j u^j (1-p)^{n-j} d^{n-j}.$$

This is the factor by which the discounted expected value of contingent receipt of the stock exceeds the current value of the stock.

6 Black-Scholes Probability Assumptions

This and the following sections do not carry the main argument any further but rather describes the foundations of the calculations performed above in Section 3.

In the Black-Scholes model, the stock price S_t at time t follows a lognormal distribution. Specifically, given the stock price S at time zero,

$$\log S_T \sim N(\log S + (\mu - \sigma^2/2)t, \sigma^2 t),$$

where μ and σ^2 are constant parameters.

In order to be precise about the interpretation of μ and σ^2, I shall need to make a few observations about the rate of return to the stock.

The return relative over the interval $[0, t]$ is S_t/S. It is lognormally distributed. Since its logarithm is normally distributed with mean $(\mu - \sigma^2/2)t$ and variance $\sigma^2 t$, it has expectation

$$\begin{aligned} E\frac{S_t}{S} &= \exp\left\{(\mu - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma^2 t\right\} \\ &= \exp(\mu t). \end{aligned}$$

The logarithm of the expected return relative is μt. So, the precise interpretation of μ is this: it is the logarithm of the expected return relative over a period of length one.

The continuously compounded rate of return per unit of time over the interval $[0, t]$ is

$$\frac{\log S_t - \log S}{t}$$

Given the current stock price S, this rate follows the normal distribution

$$N(\mu - \sigma^2/2, \sigma^2/t)$$

Thus, the precise meaning of σ^2 is this: it is the variance of the continuously compounded rate of return over a time interval of length one. The mean of this return is not μ but $\mu - \sigma^2/2$.

Note that the variance of the continuously compounded rate of return over a time interval whose length t differs from one is not σ^2 but σ^2/t. The variance depends on the length of the interval, it increases to infinity as that length decreases to zero, and it decreases toward zero as the interval length increases toward infinity. On the other hand, the mean of the rate of return is independent of the length of the time interval; but it equals $\mu - \sigma^2/2$ and not μ.

7 Risk-Adjusted Probabilities

Risk adjustment of the probabilities in this model consists in replacing μ by r, the riskless interest rate. The risk-adjusted probability distribution is such that S_t is still lognormally distributed, but the mean and variance of the normally distributed variable $\log S_t$ are now $\log S + (r - \sigma^2/2)t$ and $\sigma^2 t$, respectively. So, under the risk-adjusted probability distribution, the continuously compounded rate of return to the stock over a time interval of length one has variance σ^2, as before. The logarithm of the expected return relative over a period of length one is now r, whereas it was μ under the original probability distribution.

The risk-adjusted probability distribution has the property that the current value of any stock-price contingent claim equals the risklessly discounted value of the expected future payoff, when the expected payoff is computed using the adjusted probabilities.

This principle leads to a conceptually simple and intuitive derivation of the Black-Scholes formula. Valuing the call option and its components is just a matter of computing the expected value of some functions of a normally distributed variable (the normally distributed random variable being $\log S_T$).

8 Computing the Call Value

The value of the European call option is its discounted expected future payoff, where the expected future payoff is computed using the risk-adjusted probability, and discounting is done at the riskless rate. I shall perform this calculation for each of the component claims separately.

To compute the value of the first component claim, define a random variable Z by standardizing the variable $\log(S_T/S)$:

$$Z = \frac{\log(S_T/S) - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Under the risk-adjusted probabilities, Z follows a standard normal distribution. Define d_2 by

$$d_2 = -\frac{\log(X/S) - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

The option will be exercised if $S_T \geq X$, which is equivalent to $Z \geq -d_2$, so the risk-adjusted probability of this event is $1 - N(-d_2) = N(d_2)$. The expected exercise payment is $XN(d_2)$, and the discounted value is $\exp(-r\tau)XN(d_2)$. So that is the current value of the first component claim.

To compute the current value of the second component claim, we need the following mathematical formula.

Formula: If $\tilde{S}$ is a lognormally distributed random variable such that $\log \tilde{S} \sim N(m, s^2)$, and if X is a number, then the expectation of the truncated lognormal variable

$$\hat{S} = \begin{cases} S & \text{if } S \geq X \\ 0 & \text{otherwise} \end{cases}$$

is

$$E\hat{S} = \exp(m + s^2/2)N(s - D),$$

where

$$D = \frac{\log X - m}{s}.$$

Proof of formula:

$$
\begin{aligned}
E\hat{S} &= \frac{1}{\sqrt{2\pi}} \int_D^\infty \exp(sv+m)\exp(-v^2/2)dv \\
&= \frac{\exp(m+s^2/2)}{\sqrt{2\pi}} \int_D^\infty \exp(-(v-s)^2/2)dv \\
&= \frac{\exp(m+s^2/2)}{\sqrt{2\pi}} \int_{D-s}^\infty \exp(-y^2/2)dy \\
&= \exp(m+s^2/2)(1-N(D-s)) \\
&= \exp(m+s^2/2)N(s-D).
\end{aligned}
$$

End of proof.

Now, S_T is lognormally distributed, and $\log S_T$ has mean

$$m = \log S + (r - \frac{1}{2}\sigma^2)\tau$$

and standard deviation

$$s = \sigma\sqrt{\tau}.$$

So,

$$m + \frac{1}{2}s^2 = \log S + r\tau,$$

$$D = \frac{\log X - \log S - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = -d_2,$$

$$s - D = \sigma\sqrt{\tau} + d_2 = d_1,$$

and the expected value of S_T truncated at X is

$$
\begin{aligned}
\exp(m+s^2/2)N(s-D) &= \exp(\log S + r\tau)N(d_1) \\
&= \exp(r\tau)SN(d_1).
\end{aligned}
$$

The present discounted value is $SN(d_1)$.

9 Conclusion

I have used a derivation the Black-Scholes formula from the risk-adjusted lognormal probability distribution to arrive at interpretations of the factors $N(d_1)$ and $N(d_2)$. The first of these is the factor by which the present value of contingent receipt of the stock, contingent on exercise, exceeds the current value of the stock. The second factor is the risk-adjusted probability of exercise. The one-period and multi-period binomial formulas for the option price can be restated in such a way that they involve factors analogous to $N(d_1)$ and $N(d_2)$. While $N(d_1)$ equals the hedge ratio in the Black-Scholes model, the analogous factor in the binomial model is not equal to the hedge ratio.

10 References

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